

GENERALIZED UNSOLID AND GENERALIZED FLUID VARIETIES OF ALGEBRAS

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Abstract

Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. Strong hyperidentities are identities, which are closed under these generalized hypersubstitutions and a strongly solid variety is a variety, which every its identity is a strong hyperidentity. In this paper, we defined relations $\sim_{VRG}$, $\sim_{VRG-iso}$ on the set of all regular generalized hypersubstitutions, and investigate some algebraic structural properties of the monoid of all regular generalized hypersubstitutions.

1. Introduction

The concept of a hypersubstitution was introduced by Denecke et al. in 1991 [2]. They used it to make precise the concept of a hyperidentity. A hypersubstitution of type $\tau = (n_i)_{i \in I}$ is a mapping

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$$\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X),$$

which maps n_i -ary operation symbols to n_i -ary terms. Let $\text{Hyp}(\tau)$ be the set of all hypersubstitutions of type τ . For every $\sigma \in \text{Hyp}(\tau)$, induces a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ as follows: for any $t \in W_\tau(X)$, $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] := x$, for any variable $x \in X$, and

(ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$

are already defined.

It turns out that $(\text{Hyp}(\tau); \circ_h, \sigma_{id})$ is a monoid, where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, $\circ$ denotes the usual composition of mappings, and $\sigma_{id}(f_i) = f_i(x_1, \dots, x_{n_i})$ is the identity element.

In 1998, Chajda et al. introduced the concept of algebras induced by hypersubstitutions [1]. In 2009, Phuapong and Leeratanavalee introduced the concept of generalized derived algebras, generalized induced algebras, and studied its properties [7]. Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type $\tau = (n_i)_{i \in I}$, where f_i is an n_i -ary, $n_i \in \mathbb{N}$. Let $W_\tau(X)$ be the set of all terms of type τ built up by these operation symbols and by variables of an alphabet $X := \{x_1, x_2, x_3, \dots\}$. If $X_n := \{x_1, x_2, \dots, x_n\}$ is an n element alphabet, we obtain the set $W_\tau(X_n)$ of all n -ary terms. By $\mathcal{F}_\tau(X_n) := (W_\tau(X_n); (\tilde{f}_i)_{i \in I})$, we denote the absolutely free algebra of type τ , freely generated by $\{x_1, x_2, \dots, x_n\}$, where the operations $\tilde{f}_i = f_i^{W_\tau(X_n)}$ are defined by

$$\tilde{f}_i : (w_1, \dots, w_{n_i}) \mapsto f_i(w_1, \dots, w_{n_i}),$$

for each $i \in I$ and for all $w_1, \dots, w_{n_i} \in W_\tau(X_n)$, and $\mathcal{F}_\tau(X)$ is the absolutely free algebra of type τ , freely generated by X .

Let V be a variety of type τ and let $\underline{A} = (A; (f_i^A)_{i \in I})$ be an indexed algebra of type τ , where f_i^A is an n_i -ary operation on A . By $\text{Id}V$, we denote the set of all identities satisfied in V .

The concept of generalized hypersubstitutions was introduced by Leeratanavalee and Denecke [4]. Leeratanavalee and Denecke used it as a tool to study strong hyperidentities, and used strong hyperidentities to classify varieties into collections called *strong hypervarieties*. Varieties, which every its identity is closed under arbitrary application of generalized hypersubstitutions are called *strongly solid*.

A generalized hypersubstitution of type τ , or for short simply a generalized hypersubstitution, is a mapping σ , which maps each n_i -ary operation symbol of type τ to a term of this type in $W_\tau(X)$, which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $\text{Hyp}_G(\tau)$. To define a binary operation on $\text{Hyp}_G(\tau)$, at first, we defined inductively the concept of *superposition of terms* $S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$,

(ii) if $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)).$$

Instead of $S^m(t, t_1, \dots, t_m)$, we also write $t(t_1, \dots, t_m)$.

An algebra with similar properties can be obtained, if we define a superposition operation for n -ary operations on a set A . We consider the set $O(A)$ of all finitary operations on A .

Definition 1.1. Let $O^n(A)$, $n \geq 1$ be the set of all n -ary operations defined on the set A . Then, $(n+1)$ -ary superposition operation $S^{n,A} : O^n(A)^{n+1} \rightarrow O^n(A)$ is defined by $S^{n,A}(f^A, g_1^A, \dots, g_n^A)(a_1, \dots, a_n) := f^A(g_1^A(a_1, \dots, a_n), \dots, g_n^A(a_1, \dots, a_n))$, for every $(a_1, \dots, a_n) \in A^n$.

Here $g_1^A, \dots, g_n^A$ as well as are n -ary. This can be generalized to an operation $S_m^{n,A} : O^n(A) \times (O^m(A))^n \rightarrow O^m(A)$, $m \geq 1$ defined by

$$S_m^{n,A}(f^A, g_1^A, \dots, g_n^A)(a_1, \dots, a_m) := f^A(g_1^A(a_1, \dots, a_m), \dots, g_n^A(a_1, \dots, a_m)).$$

Definition 1.2. Let $\underline{A}$ be an algebra of type $\tau = (n_i)_{i \in I}$. Let $t \in W_\tau(X)$. Then, t induces an n -ary term operation $t^A : A^n \rightarrow A$ called the *n -ary term operation induced by the term t* on the algebra $\underline{A}$, via the following steps:

- (i) if $t = x_j \in X_n$, then $t^A = x_j^A := e_j^{n,A}$, where $e_j^{n,A} : (a_1, \dots, a_n) \mapsto a_j$ is n -ary projection onto the j -th coordinate,
- (ii) if $t = x_j \in X \setminus X_n$, then $t^A = x_j^A := c_a^n$ is the n -ary constant operation on A with value a , and each element from A is uniquely induced by an element from $X \setminus X_n$,
- (iii) if $t = f_i(t_1, \dots, t_{n_i})$ and $t_1^A, \dots, t_{n_i}^A$ are the n -ary term operations, which are induced by $t_1, \dots, t_{n_i}$, then $t^A = S^{n_i,A}(f_i^A, t_1^A, \dots, t_{n_i}^A)$.

We extended a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ inductively defined as follows:

- (i) $\hat{\sigma}[x] := x \in X$,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i -ary operation symbol f_i , where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then, we defined a binary operation $\circ_G$ on $\text{Hyp}_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ denotes the usual composition of mappings. Let σ_{id} be a hypersubstitution, which maps each n_i -ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. Leeratanavalee and Denecke proved the following propositions.

Proposition 1.3 [4]. *For arbitrary terms $t, t_1, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$, we have*

$$(i) S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)],$$

$$(ii) (\hat{\sigma}_1 \circ \sigma_2)^\wedge = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Proposition 1.4 [4]. *$\text{Hyp}_G(\tau) = (\text{Hyp}_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $\text{Hyp}(\tau) = (\text{Hyp}(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $\text{Hyp}_G(\tau)$.*

An identity $u \approx v$ is called an M -strong hyperidentity of a variety V , where $\underline{M}$ is any submonoid of $\text{Hyp}_G(\tau)$, if for any generalized hypersubstitution $\sigma \in \underline{M}$, the identity $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in V . A variety V is called an M -strongly solid, if every identity of V is an M -strong hyperidentity of V . If $\underline{M}$ is the monoid $\text{Hyp}_G(\tau)$, an M -strong hyperidentity is called a *strong hyperidentity*, and an M -strongly solid variety is called a *strongly solid variety*.

Generalized hypersubstitution can also be applied to algebras. For an algebra $\underline{A} = (A; (f_i^A)_{i \in I})$ of type τ , we define the *generalized derived algebra* $\sigma[\underline{A}]$ by $\sigma[\underline{A}] = (A; (\sigma(f_i)^A)_{i \in I})$. The generalized derived algebra $\sigma[\underline{A}]$ is an algebra of the same type τ and with the same universe A . The set $\{\sigma(f_i)^A \mid i \in I\}$ of fundamental operations of $\sigma[\underline{A}]$ is a subset of the set of all term operations of the algebra $\underline{A}$. If V is a strongly solid variety, then $\sigma(V) := \{\sigma[\underline{A}] \mid \underline{A} \in V\} \subseteq V$, for all $\sigma \in \text{Hyp}_G(\tau)$.

2. Regular Generalized Hypersubstitutions and M -Strongly Solid Varieties

In this section, we recall some basic facts about regular generalized hypersubstitutions and M -strongly solid varieties.

Definition 2.1 [6]. A generalized hypersubstitution $\sigma \in \text{Hyp}_G(\tau)$ is called a *regular generalized hypersubstitution*, if for every $i \in I$, each of the variables $x_1, x_2, \dots, x_{n_i}$ occur in $\hat{\sigma}[f_i(x_1, x_2, \dots, x_{n_i})]$.

Let $\text{Reg}_G(\tau)$ be the set of all regular generalized hypersubstitutions of type τ . Then, we have $\text{Reg}_G(\tau) \subseteq \text{Hyp}_G(\tau)$.

Proposition 2.2 [6]. *For any type τ , $\text{Reg}_G(\tau)$ is a submonoid of $\text{Hyp}_G(\tau)$.*

Lemma 2.3. *Let $\underline{A}$ be an algebra of type $\tau = (n_i)_{i \in I}$; $n_i \in \mathbb{N}$. For $\sigma \in \text{Reg}_G(\tau)$ and $t \in W_\tau(X)$, we denoted the n -ary term operation induced by t in the algebra $\sigma[\underline{A}]$, and the n -ary term operation induced by $\hat{\sigma}[t]$ in the algebra $\underline{A}$ by $t^{\sigma[\underline{A}]}$ and $\hat{\sigma}[t]^A$, respectively. Then, we have*

$$\hat{\sigma}[t]^A = t^{\sigma[\underline{A}]}.$$

Proof. We will proceed by induction on the complexity of the term t .

If $t = x_i \in X_n$, then $\hat{\sigma}[x_i]^A = x_i^A = e_i^{n,A} = x_i^{\sigma[\underline{A}]}$.

If $t = x_j \in X \setminus X_n$, then $\hat{\sigma}[x_j]^A = x_j^A := c_a^n = x_j^{\sigma[\underline{A}]}$ is the n -ary constant operation on $\underline{A}$ with value a , and every element from A is uniquely induced by an element from $X \setminus X_n$. If $t = f_i(t_1, \dots, t_{n_i})$ and assume that $\hat{\sigma}[t_j]^A = t_j^{\sigma[\underline{A}]}$, for all $1 \leq j \leq n_i$, then

$$\begin{aligned} \hat{\sigma}[f_i(t_1, \dots, t_{n_i})]^A &= S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])^A \\ &= S^{n_i,A}(\sigma(f_i)^A, \hat{\sigma}[t_1]^A, \dots, \hat{\sigma}[t_{n_i}]^A) \\ &= S^{n_i,A}(f_i^{\sigma[\underline{A}]}, t_1^{\sigma[\underline{A}]}, \dots, t_{n_i}^{\sigma[\underline{A}]}) \\ &= f_i(t_1, \dots, t_{n_i})^{\sigma[\underline{A}]}. \end{aligned}$$

□

Lemma 2.4. *Let $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$ and $\underline{A} \in \text{Alg}(\tau)$. Then $\sigma_1(\sigma_2[\underline{A}]) = (\sigma_2 \circ_G \sigma_1)[\underline{A}]$.*

Proof. We have

$$\begin{aligned} \sigma_1(\sigma_2[\underline{A}]) &= (A; (\sigma_1(f_i)^{\sigma_2[\underline{A}]})_{i \in I}) \\ &= (A; (\hat{\sigma}_2[\sigma_1(f_i)]^A)_{i \in I}) \text{ by Lemma 2.3} \\ &= (A; ((\sigma_2 \circ_G \sigma_1)(f_i)^A)_{i \in I}) \\ &= (\sigma_2 \circ_G \sigma_1)[\underline{A}]. \end{aligned}$$

□

Let $K \subseteq \text{Alg}(\tau)$ be a class of algebras of type τ and $\sum \subseteq W_\tau(X)^2$. Consider the connection between $\text{Alg}(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$Id : \mathcal{P}(\text{Alg}(\tau)) \rightarrow \mathcal{P}(W_\tau(X)^2), \quad Mod : \mathcal{P}(W_\tau(X)^2) \rightarrow \mathcal{P}(\text{Alg}(\tau)),$$

with

$$\begin{aligned} IdK &:= \{s \approx t \in W_\tau(X)^2 \mid \forall \underline{A} \in K (\underline{A} \models s \approx t)\}, \\ Mod \sum &:= \{\underline{A} \in \text{Alg}(\tau) \mid \forall s \approx t \in \sum (\underline{A} \models s \approx t)\}. \end{aligned}$$

Clearly, the pair (Mod, Id) is a Galois connection between $\text{Alg}(\tau)$ and $W_\tau(X)^2$. We have two closure operators $ModId$ and $IdMod$, and their sets of fixed points.

Definition 2.5. Let $\underline{A}$ be an algebra of type τ and let $\underline{M}$ be a submonoid of the monoid $\text{Reg}_G(\tau)$. Then, we define

$$\chi_M^A : \mathcal{P}(\text{Alg}(\tau)) \rightarrow \mathcal{P}(\text{Alg}(\tau)), \quad \chi_M^E : \mathcal{P}(W_\tau(X)^2) \rightarrow \mathcal{P}(W_\tau(X)^2),$$

by

$$\chi_M^A(\underline{A}) := \{\sigma[\underline{A}] \mid \sigma \in M\},$$

$$\chi_M^E[s \approx t] := \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in M\}.$$

For $K \subseteq \text{Alg}(\tau)$ and $\sum \subseteq W_\tau(X)^2$, we define $\chi_M^A(K) := \bigcup_{A \in K} \chi_M^A(\underline{A})$ and

$$\chi_M^E[\sum] := \bigcup_{s \approx t \in \sum} \chi_M^E[s \approx t].$$

Proposition 2.6. *Let $\underline{A} \in \text{Alg}(\tau)$ and $s \approx t \in W_\tau(X)^2$. Then*

$$\chi_M^A(\underline{A}) \models s \approx t \text{ iff } \underline{A} \models \chi_M^E[s \approx t].$$

Proof. We have

$$\begin{aligned} \chi_M^A(\underline{A}) \models s \approx t &\Leftrightarrow \forall \sigma \in M(\sigma[\underline{A}] \models s \approx t) \\ &\Leftrightarrow \forall \sigma \in M(s^{\sigma[\underline{A}]} = t^{\sigma[\underline{A}]}) \\ &\Leftrightarrow \forall \sigma \in M(\hat{\sigma}[s]^A = \hat{\sigma}[t]^A) \\ &\Leftrightarrow \forall \sigma \in M(\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t]) \\ &\Leftrightarrow \underline{A} \models \chi_M^E[s \approx t]. \end{aligned}$$

□

Definition 2.7. Let V be a variety of algebras of type τ . Then V is said to be *M-strongly solid*, if $\chi_M^A(V) = V$. If $M = \text{Reg}_G(\tau)$, then V is called *regular strongly solid*.

3. V-Proper Regular Generalized Hypersubstitutions

In this section, we give the definition of V-proper regular generalized hypersubstitution, V-regular generalized equivalent, and investigate some algebraic structural properties of the set of all V-proper regular generalized hypersubstitutions and V-regular generalized equivalent.

Definition 3.1. Let V be a variety of algebras of type τ . A regular generalized hypersubstitution $\sigma \in \text{Reg}_G(\tau)$ is called a *V-proper regular generalized hypersubstitution*, if for every $s \approx t \in \text{Id}V$ one gets $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id}V$.

We denote $P_{RG}(V)$ for the set of all V -proper regular generalized hypersubstitutions of type τ .

Proposition 3.2. *The algebra $(P_{RG}(V); \circ_G, \sigma_{id})$ is a submonoid of $(\text{Reg}_G(\tau); \circ_G, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in P_{RG}(V)$. If $\sigma_1, \sigma_2 \in P_{RG}(V)$, then for every $s \approx t \in \text{Id}V$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in \text{Id}V$ and $\hat{\sigma}_1[\hat{\sigma}_2[s]] \approx \hat{\sigma}_1[\hat{\sigma}_2[t]] \in \text{Id}V$. This means that $(\hat{\sigma}_1 \circ \hat{\sigma}_2)[s] \approx (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t] \in \text{Id}V$ and thus, $(\sigma_1 \circ_G \sigma_2)[s] \approx (\sigma_1 \circ_G \sigma_2)[t] \in \text{Id}V$. Therefore, $\sigma_1 \circ_G \sigma_2 \in P_{RG}(V)$, and then $P_{RG}(V)$ is a submonoid of $\text{Reg}(\tau)$. $\square$

Definition 3.3. Let V be a variety of algebras of type τ . Two regular generalized hypersubstitutions σ_1, σ_2 of type τ are called *V-regular generalized equivalent*, if and only if $\sigma_1(f_i) \approx \sigma_2(f_i) \in \text{Id}V$, for all $i \in I$. In this case, we write $\sigma_1 \sim_{VRG} \sigma_2$.

Theorem 3.4. *Let V be a variety of algebras of type τ , and let $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$. Then, the following are equivalent:*

- (i) $\sigma_1 \sim_{VRG} \sigma_2$.
- (ii) For every $t \in W_\tau(X)$, the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in \text{Id}V$.
- (iii) For every $\underline{A} \in V$, $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$, where $\sigma_k[\underline{A}] = (A; (\sigma_k(f_i)^A)_{i \in I})$; $k = 1, 2$.

Proof. (i) $\Rightarrow$ (ii). We will proceed by induction on the complexity of the term t .

If $t = x_i \in X$, $i \in \mathbb{N}$, then $\hat{\sigma}_1[x_i] = x_i \approx x_i = \hat{\sigma}_2[x_i] \in IdV$.

If $t = f_i(t_1, \dots, t_{n_i})$ and assume that $\hat{\sigma}_1[t_j] \approx \hat{\sigma}_2[t_j] \in IdV$, for all $j \in \{1, 2, \dots, n_i\}$, then

$$\begin{aligned} \hat{\sigma}_1[f_i(t_1, \dots, t_{n_i})]^A &= S^{n_i}(\sigma_1(f_i), \hat{\sigma}_1[t_1], \dots, \hat{\sigma}_1[t_{n_i}])^A \\ &= S^{n_i, A}(\sigma_1(f_i)^A, \hat{\sigma}_1[t_1]^A, \dots, \hat{\sigma}_1[t_{n_i}]^A) \\ &= S^{n_i, A}(\sigma_2(f_i)^A, \hat{\sigma}_2[t_1]^A, \dots, \hat{\sigma}_2[t_{n_i}]^A) \\ &= S^{n_i}(\sigma_2(f_i), \hat{\sigma}_2[t_1], \dots, \hat{\sigma}_2[t_{n_i}])^A \\ &= \hat{\sigma}_2[f_i(t_1, \dots, t_{n_i})]^A. \end{aligned}$$

Hence, $\hat{\sigma}_1[f_i(t_1, \dots, t_{n_i})] \approx \hat{\sigma}_2[f_i(t_1, \dots, t_{n_i})] \in IdV$.

(ii) $\Rightarrow$ (iii). Let $\underline{A} \in V$. From (ii), we have $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$ for all $t \in W_\tau(X)$. Then $\underline{A} \models \hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$. Thus, $\sigma_1(f_i)^A = \sigma_2(f_i)^A$ for all $i \in I$. Therefore, $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$.

(iii) $\Rightarrow$ (i). Let $\underline{A} \in V$ and $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$. Then $\sigma_1(f_i)^A = \sigma_2(f_i)^A$ for all $i \in I$. Thus, $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$. So, $\sigma_1 \sim_{VRG} \sigma_2$. $\square$

Proposition 3.5. *Let V be a solid variety of algebras of type τ . Then, the relation $\sim_{VRG}$ is a congruence on $\text{Reg}_G(\tau)$.*

Proof. It is clear that $\sim_{VRG}$ is an equivalence relation on $\text{Reg}_G(\tau)$. Assume that $\sigma_1 \sim_{VRG} \sigma_2$ and $\sigma_3 \sim_{VRG} \sigma_4$. Then, by Theorem 3.4 (ii), for all $i \in I$, we get

$$\hat{\sigma}_1[\sigma_3(f_i)] \approx \hat{\sigma}_2[\sigma_3(f_i)] \in IdV \Rightarrow \sigma_1 \circ_G \sigma_3 \approx \sigma_2 \circ_G \sigma_3 \in IdV,$$

and

$$\hat{\sigma}_2[\sigma_3(f_i)] \approx \hat{\sigma}_2[\sigma_4(f_i)] \in IdV \Rightarrow \sigma_2 \circ_G \sigma_3 \approx \sigma_2 \circ_G \sigma_4 \in IdV.$$

By transitivity, we get $\sigma_1 \circ_G \sigma_3 \approx \sigma_2 \circ_G \sigma_4 \in IdV$, and so, $\sigma_1 \circ_G \sigma_3 \sim_{VRG} \sigma_2 \circ_G \sigma_4$. $\square$

Proposition 3.6. *Let V be a variety of algebras of type τ . Then the following hold:*

(i) *For all $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$, if $\sigma_1 \sim_{VRG} \sigma_2$, then σ_1 is a V -proper regular generalized hypersubstitution, iff σ_2 is a V -proper regular generalized hypersubstitution.*

(ii) *For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$, if $\sigma_1 \sim_{VRG} \sigma_2$, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

Proof. (i) Let σ_1 be a V -proper regular generalized hypersubstitution. Then for all $s \approx t \in IdV$, the equation $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$. Since $\sigma_1 \sim_{VRG} \sigma_2$, we have $V \models \hat{\sigma}_2[s] \approx \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$. Thus, σ_2 is a V -proper regular generalized hypersubstitution. The other direction can be proved in the same way.

(ii) Suppose that $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$. Since $\sigma_1 \sim_{VRG} \sigma_2$, the equations $\hat{\sigma}_1[s] \approx \hat{\sigma}_2[s]$ and $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in V by Theorem 3.4. Thus, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$. The converse can be proved in the same way. $\square$

Corollary 3.7. *The set $P_{RG}(V)$ is a union of equivalence classes with respect to $\sim_{VRG}$.*

Now, we consider the equivalence class of the identity generalized hypersubstitution.

Definition 3.8. Let V be a variety of algebras of type τ . A regular generalized hypersubstitution $\sigma \in \text{Reg}_G(\tau)$ is called an *inner regular generalized hypersubstitution* of a variety V , if for every $i \in I$,

$$\hat{\sigma}[f_i(x_1, \dots, x_{n_i})] \approx f_i(x_1, \dots, x_{n_i}) \in IdV.$$

Let $P_{RG}^0(V)$ be the set of all inner regular generalized hypersubstitutions of V . By the definition, $P_{RG}^0(V)$ is the equivalence class $[\sigma_{id}]_{\sim_{VRG}}$.

Proposition 3.9. *If $\sigma \in P_{RG}^0(V)$, then $\hat{\sigma}[t] \approx t \in IdV$, for all $t \in W_\tau(X)$.*

Proof. We will proceed by induction on the complexity of the term t .

If $t = x_i \in X$, then $\hat{\sigma}[t] = x_i \approx x_i = t \in IdV$.

If $t = f_i(t_1, \dots, t_{n_i})$ and assume that $\hat{\sigma}[t_j] \approx t_j \in IdV$, for all $j \in \{1, 2, \dots, n_i\}$, then

$$\begin{aligned} \hat{\sigma}[f_i(t_1, \dots, t_{n_i})]^A &= S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])^A \\ &= S^{n_i, A}(\sigma(f_i)^A, \hat{\sigma}[t_1]^A, \dots, \hat{\sigma}[t_{n_i}]^A) \\ &= S^{n_i, A}(f_i^A, t_1^A, \dots, t_{n_i}^A) \\ &= f_i(t_1, \dots, t_{n_i})^A, \quad \text{for all } \underline{A} \in V. \end{aligned}$$

□

Proposition 3.10. *The algebra $(P_{RG}^0(V); \circ_G, \sigma_{id})$ is a submonoid of $(P_{RG}(V); \circ_G, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in P_{RG}^0(V)$. Assume that $\sigma_1, \sigma_2 \in P_{RG}^0(V)$. Then

$$\begin{aligned} (\sigma_1 \circ_G \sigma_2)[f_i(x_1, \dots, x_{n_i})] &= \hat{\sigma}_1[\hat{\sigma}_2[f_i(x_1, \dots, x_{n_i})]] \\ &\approx \hat{\sigma}_2[f_i(x_1, \dots, x_{n_i})] \quad \text{by Proposition 3.9} \\ &\approx f_i(x_1, \dots, x_{n_i}) \quad \text{by Proposition 3.9} \\ &\in IdV. \end{aligned}$$

Therefore, $\sigma_1 \circ_G \sigma_2 \in P_{RG}^0(V)$. Thus $\underline{P_{RG}^0(V)}$ is a monoid. By Proposition 3.9, we have $P_{RG}^0(V) \subseteq P_{RG}(V)$. So, $(P_{RG}^0(V); \circ_G, \sigma_{id})$ is a submonoid of $(P_{RG}(V); \circ_G, \sigma_{id})$. □

Now, we show that the compatibility condition from the definition of a homomorphism for algebras transfers fundamental operations to arbitrary term operations.

Lemma 3.11. *Let $\underline{A} \in \text{Alg}(\tau)$ and let t^A be the n -ary term operation on A induced by the n -ary term $t \in W_\tau(X)$. Let $\underline{B} \in \text{Alg}(\tau)$. If $\varphi : \underline{A} \rightarrow \underline{B}$ is an isomorphism, then for all $a_1, \dots, a_{n_i} \in A$,*

$$\varphi(t^A(a_1, \dots, a_{n_i})) = t^B(\varphi(a_1), \dots, \varphi(a_{n_i})).$$

Proof. We will proceed by induction on the complexity of the term t .

If $t = x_i \in X_n$, then $t^A = x_i^A = e_i^{n,A}$, and $t^B = x_i^B = e_i^{n,B}$. Thus,

$$\begin{aligned} \varphi(t^A(a_1, \dots, a_{n_i})) &= \varphi(e_i^{n,A}(a_1, \dots, a_{n_i})) \\ &= \varphi(a_i) \\ &= e_i^{n,B}(\varphi(a_1), \dots, \varphi(a_{n_i})) \\ &= t^B(\varphi(a_1), \dots, \varphi(a_{n_i})). \end{aligned}$$

If $t = x_j \in X \setminus X_n$, then $t^A = x_j^A = c_a^n$ is the n -ary constant operation on A with value a , and each element from A is uniquely induced by an element from $X \setminus X_n$, and $t^B = x_j^B = c_b^n$ is the n -ary constant operation on B with value b , and each element from B is uniquely induced by an element from $X \setminus X_n$. Thus,

$$\begin{aligned} \varphi(t^A(a_1, \dots, a_{n_i})) &= \varphi(x_j^A(a_1, \dots, a_{n_i})) \\ &= \varphi(a) \\ &= b, \text{ for some unique } b \in B \\ &= x_j^B(\varphi(a_1), \dots, \varphi(a_{n_i})) \\ &= t^B(\varphi(a_1), \dots, \varphi(a_{n_i})). \end{aligned}$$

If $t = f_i(t_1, \dots, t_{n_i})$, and assume that $\varphi(t_j^A(a_1, \dots, a_{n_i})) = t_j^B(\varphi(a_1), \dots, \varphi(a_{n_i}))$, for all $1 \leq j \leq n_i$, then

$$\begin{aligned}
\varphi(t^A(a_1, \dots, a_{n_i})) &= \varphi(f_i(t_1, \dots, t_{n_i})^A(a_1, \dots, a_{n_i})) \\
&= \varphi(S^{n_i, A}(f_i^A, t_1^A, \dots, t_{n_i}^A)(a_1, \dots, a_{n_i})) \\
&= \varphi(f_i^A, t_1^A(a_1, \dots, a_{n_i}), \dots, t_{n_i}^A(a_1, \dots, a_{n_i})) \\
&= f_i^B(\varphi(t_1^A(a_1, \dots, a_{n_i})), \dots, \varphi(t_{n_i}^A(a_1, \dots, a_{n_i}))) \\
&= f_i^B(t_1^B(\varphi(a_1), \dots, \varphi(a_{n_i})), \dots, t_{n_i}^B(\varphi(a_1), \dots, \varphi(a_{n_i}))) \\
&= f_i^B((t_1^B, \dots, t_{n_i}^B)(\varphi(a_1), \dots, \varphi(a_{n_i}))) \\
&= t^B(\varphi(a_1), \dots, \varphi(a_{n_i})). \quad \square
\end{aligned}$$

Proposition 3.12. *Let $\underline{A}, \underline{B} \in \text{Alg}(\tau)$ and $\sigma \in \text{Reg}_G(\tau)$. If $h : \underline{A} \rightarrow \underline{B}$ is an isomorphism, then h is also an isomorphism from $\sigma[\underline{A}]$ to $\sigma[\underline{B}]$.*

Proof. Since $h : A \rightarrow B$ is bijective, the mapping $h : \sigma[\underline{A}] \rightarrow \sigma[\underline{B}]$ is also bijective because algebras and their derived algebras have the same universes. By Lemma 3.11, for the term $\sigma(f_i)$, we have

$$\begin{aligned}
h(f_i^{\sigma[\underline{A}]}(a_1, \dots, a_{n_i})) &= h(\sigma(f_i)^A(a_1, \dots, a_{n_i})) \\
&= \sigma(f_i)^B(h(a_1), \dots, h(a_{n_i})) \\
&= f_i^{\sigma[\underline{B}]}(h(a_1), \dots, h(a_{n_i})).
\end{aligned}$$

This shows that $h : A \rightarrow B$ is a homomorphism. Therefore, $\sigma[\underline{A}] \cong \sigma[\underline{B}]$. $\square$

Definition 3.13. Let V be a variety of algebras of type τ and $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$. Then, we define

$$\sigma_1 \sim_{VRG-iso} \sigma_2 \text{ iff } \sigma_1[\underline{A}] \cong \sigma_2[\underline{A}], \forall \underline{A} \in V.$$

Clearly, $\sim_{VRG} \subseteq \sim_{VRG-iso}$.

Proposition 3.14. Let V be a variety of algebras of type τ . Then,

- (i) the relation $\sim_{VRG-iso}$ is a right congruence on $\text{Reg}_G(\tau)$;
- (ii) if V is a strongly solid variety, then $\sim_{VRG-iso}$ is a congruence on $\text{Reg}_G(\tau)$.

Proof. (i) Let $\sigma_1 \sim_{VRG-iso} \sigma_2$ and $\sigma \in \text{Reg}_G(\tau)$. Then $\sigma_1[\underline{A}] \cong \sigma_2[\underline{A}]$ and $\sigma[\sigma_1[\underline{A}]] \cong \sigma[\sigma_2[\underline{A}]]$, for all $\underline{A} \in V$ by Lemma 3.12. Consider $(\sigma_1 \circ_G \sigma)[\underline{A}] = \sigma[\sigma_1[\underline{A}]] \cong \sigma[\sigma_2[\underline{A}]] = (\sigma_2 \circ_G \sigma)[\underline{A}]$. So, $\sigma_1 \circ_G \sigma \sim_{VRG-iso} \sigma_2 \circ_G \sigma$. This shows that $\sim_{VRG-iso}$ is a right congruence.

(ii) Assume that V is strongly solid. Then, $\sigma[\underline{A}] \in V$ for all $\sigma \in \text{Reg}_G(\tau)$ for all $\underline{A} \in V$. Since $\sigma_1 \sim_{VRG-iso} \sigma_2$, $\sigma_1[\sigma[\underline{A}]] \cong \sigma_2[\sigma[\underline{A}]]$ for all $\underline{A} \in V$. Consider $(\sigma \circ_G \sigma_1)[\underline{A}] = \sigma_1[\sigma[\underline{A}]] \cong \sigma_2[\sigma[\underline{A}]] = (\sigma \circ_G \sigma_2)[\underline{A}]$. So, $\sigma \circ_G \sigma_1 \sim_{VRG-iso} \sigma \circ_G \sigma_2$. This shows that $\sim_{VRG-iso}$ is a left congruence and because of (i), it is a congruence. $\square$

Proposition 3.15. If $V = \text{Alg}(\tau)$, then $\sim_{VRG-iso}$ is a congruence on $\text{Reg}_G(\tau)$.

Proof. Since $V = \text{Alg}(\tau)$, so V is a strongly solid variety. Thus, the claim follows from Proposition 3.14 (ii). $\square$

Proposition 3.16. The equivalence class $P_{RG}^{0, VRG-iso}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$ is the submonoid of $(\text{Reg}_G(\tau); \circ_G, \sigma_{id})$.

Proof. Clearly, $\sigma_{id} \in P_{RG}^{0, VRG-iso}(V)$. Next, we will show that $P_{RG}^{0, VRG-iso}(V)$ is closed under the operation $\circ_G$. Let $\sigma_1, \sigma_2 \in P_{RG}^{0, VRG-iso}(V)$. Then $\sigma_1 \sim_{VRG-iso} \sigma_{id}$ and $\sigma_2 \sim_{VRG-iso} \sigma_{id}$. These imply that $\sigma_1[\underline{A}] \cong \underline{A}$ and $\sigma_2[\underline{A}] \cong \underline{A}$, for all $\underline{A} \in V$. We have

$$\begin{aligned} (\sigma_1 \circ_G \sigma_2)[\underline{A}] &= \sigma_2[\sigma_1[\underline{A}]], \quad \text{by Lemma 2.4} \\ &\cong \sigma_2[\underline{A}] \\ &\cong \underline{A}. \end{aligned}$$

Then $(\sigma_1 \circ_G \sigma_2) \sim_{VRG-iso} \sigma_{id}$. Therefore, $\sigma_1 \circ_G \sigma_2 \in P_{RG}^{0, VRG-iso}(V)$. So, $P_{RG}^{0, VRG-iso}(V)$ is the submonoid of $\text{Reg}_G(\tau)$. $\square$

Proposition 3.17. *Let V be a variety of algebras of type τ , $s \approx t \in IdV$ for $s, t \in W_\tau(X)$ and let $\sigma_1, \sigma_2 \in \text{Reg}_G(\tau)$. If $\sigma_1 \sim_{VRG-iso} \sigma_2$ and $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

Proof. Assume that $\sigma_1 \sim_{VRG-iso} \sigma_2$ and $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$. Then by Theorem 3.4, we have $\sigma_1[\underline{A}] \cong \sigma_2[\underline{A}]$ for all $\underline{A} \in V$. So, there is an isomorphism φ from $\sigma_1[\underline{A}]$ to $\sigma_2[\underline{A}]$. Let $b_1, \dots, b_{n_i} \in A$. Then, there are elements $a_1, \dots, a_{n_i} \in A$, such that $\varphi(a_1) = b_1, \dots, \varphi(a_{n_i}) = b_{n_i}$. We have

$$\begin{aligned} \hat{\sigma}_2[s](b_1, \dots, b_{n_i}) &= \hat{\sigma}_2[s](\varphi(a_1), \dots, \varphi(a_{n_i})) \\ &= \varphi(\hat{\sigma}_1[s](a_1, \dots, a_{n_i})) \\ &= \varphi(\hat{\sigma}_1[t](a_1, \dots, a_{n_i})) \\ &= \hat{\sigma}_2[t](\varphi(a_1), \dots, \varphi(a_{n_i})) \\ &= \hat{\sigma}_2[t](b_1, \dots, b_{n_i}). \end{aligned}$$

Then, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\underline{A}$ for all $\underline{A} \in V$. So, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$. $\square$

Corollary 3.18. *The set $P_{RG}(V)$ is a union of equivalence classes with respect to $\sim_{VRG-iso}$.*

Remark 3.19. $P_{RG}^0(V) \subseteq P_{RG}^{0,VRG-iso}(V) \subseteq P_{RG}(V)$.

4. Generalized Unsolid and Generalized Fluid Varieties

For a strongly solid variety, every identity is closed under all generalized hypersubstitutions. At the other extreme is the case, where the identities are closed only under the identity hypersubstitution.

Definition 4.1. A variety V of algebras of type τ is said to be *generalized unsolid*, if $P_{RG}(V) = P_{RG}^0(V)$, and V is said to be *completely generalized unsolid*, if $P_{RG}(V) = P_{RG}^0(V) = \{\sigma_{id}\}$.

Definition 4.2. A variety V of algebras of type τ is said to be *iso-generalized unsolid*, if $P_{RG}(V) = P_{RG}^{0,VRG-iso}(V)$, and V is said to be *completely iso-generalized unsolid*, if $P_{RG}(V) = P_{RG}^{0,VRG-iso}(V) = \{\sigma_{id}\}$.

Proposition 4.3. *Let V be a variety of algebras of type τ . Then,*

- (i) *if V is generalized unsolid, V is iso-generalized unsolid;*
- (ii) *V is completely generalized unsolid, if and only if V is completely iso-generalized unsolid.*

Proof. (i) The claim follows from the definitions and Remark 3.19.

(ii) If V is completely generalized unsolid, then V is completely iso-generalized unsolid by Remark 3.19. Conversely, assume that V is completely iso-generalized unsolid. Then $P_{RG}(V) = P_{RG}^{0,VRG-iso}(V) = \{\sigma_{id}\}$. Since, $P_{RG}^0(V) \subseteq P_{RG}(V)$ and $P_{RG}(V) \neq \emptyset$, we get $P_{RG}^0(V) = \{\sigma_{id}\}$. So, V is completely generalized unsolid. $\square$

Definition 4.4. A variety V of algebras of type τ is said to be *generalized fluid*, if for every algebra $\underline{A} \in V$, and every regular generalized hypersubstitution $\sigma \in \text{Reg}_G(\tau)$, there holds

$$\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}] \cong \underline{A}.$$

We denote by $\sigma(V)$, the class of all algebras $\sigma[\underline{A}]$ with $\underline{A} \in V$. As an easy consequence of the definition, we have the following result.

Proposition 4.5. *If a variety V of algebras of type τ is generalized fluid, then for every $\sigma \in \text{Reg}_G(\tau)$, there holds*

$$\sigma(V) \subseteq V \Rightarrow \forall \underline{A} \in V (\sigma[\underline{A}] \cong \underline{A}).$$

Proposition 4.6. *Let V be a variety of algebras of type τ . Then for all $\sigma \in \text{Reg}_G(\tau)$, $\sigma(V) \subseteq V$, if and only if $\sigma \in P_{RG}(V)$.*

Proof. Assume that $\sigma(V) \subseteq V$. Let $s \approx t \in IdV$. Then, $IdV \subseteq Id\sigma(V)$ and we have $s \approx t \in Id\sigma(V)$. So, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ by Proposition 2.6. Therefore, $\sigma \in P_{RG}(V)$. Conversely, we assume that $\sigma \in P_{RG}(V)$. Let $\underline{A} \in \sigma(V)$ and $s \approx t \in IdV$. Then, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ by $\sigma \in P_{RG}(V)$ and $s \approx t \in Id\sigma(V)$ by Proposition 2.6. Since $\underline{A} \in \sigma(V)$, we have $s \approx t \in Id\underline{A}$ and $\underline{A} \in V$. So, $\sigma(V) \subseteq V$. $\square$

This shows that, if a variety V of algebras of type τ is generalized fluid, then for every regular generalized hypersubstitution $\sigma \in \text{Reg}_G(\tau)$, there holds

$$\sigma \in P_{RG}(V) \Rightarrow \forall \underline{A} \in V (\sigma[\underline{A}] \cong \underline{A}).$$

Proposition 4.7. *Let V be a generalized fluid variety of algebras of type τ . Then, $P_{RG}(V) = [\sigma_{id}]_{VRG-iso}$.*

Proof. Let $\sigma \in P_{RG}(V)$. Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all $s \approx t \in IdV$ implies that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\underline{A}$, for all $\underline{A} \in V$. By Proposition 2.6, we have $s \approx t \in Id\sigma[\underline{A}]$. So, $\sigma[\underline{A}] \in V$ for all $\underline{A} \in V$ and for all $\sigma \in \text{Reg}_G(\tau)$.

Since V is generalized fluid, we have $\sigma[\underline{A}] \cong \underline{A}$ and this implies that $\sigma \sim_{VRG-iso} \sigma_{id}$. Therefore, $\sigma \in [\sigma_{id}]_{\sim_{VRG-iso}}$. Thus $P_{RG}(V) \subseteq [\sigma_{id}]_{\sim_{VRG-iso}}$, but $[\sigma_{id}]_{\sim_{VRG-iso}} \subseteq P_{RG}(V)$. So, $P_{RG}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$.

□

Proposition 4.8. *Let V be a strongly solid variety of algebras of type τ . Then V is generalized fluid, if and only if $P_{RG}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$.*

Proof. By Proposition 4.7, we have that V is generalized fluid, then $P_{RG}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$. Conversely, we assume that $P_{RG}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$. Let $\sigma \in \text{Reg}_G(\tau)$. Since V is strongly solid, we get $\sigma[\underline{A}] \in V$, for all $\underline{A} \in V$. Next, we will show that $\sigma \in P_{RG}(V)$. Suppose that $\sigma \notin P_{RG}(V)$. Then, there is an identity $s \approx t \in IdV$, such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin IdV$, and this implies that there exists $\underline{A} \in V$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\underline{A}$. By Proposition 2.6, we get $s \approx t \notin Id\sigma[\underline{A}]$ and $\sigma[\underline{A}] \notin V$, which is a contradiction. So, $\sigma \in P_{RG}(V) = [\sigma_{id}]_{\sim_{VRG-iso}}$ and $\sigma \sim_{VRG-iso} \sigma_{id}$. Therefore, $\sigma[\underline{A}] \cong \underline{A}$ for all $\underline{A} \in V$. Then V is generalized fluid. □

Let V be a generalized fluid variety of algebras of type τ and assume that W is a subvariety of V . Clearly, W is also generalized fluid since, for all $\underline{A} \in W \subseteq V$ and $\sigma \in \text{Reg}_G(\tau)$, we have

$$\sigma[\underline{A}] \in W \Rightarrow \sigma[\underline{A}] \cong \underline{A}.$$

Therefore, we have the following.

Proposition 4.9. *Every subvariety of a generalized fluid variety of algebras of type τ is generalized fluid.*

Definition 4.10. A variety V of algebras of type τ is *strongly generalized fluid*, if for every regular generalized hypersubstitution $\sigma \in \text{Reg}_G(\tau)$, there holds

$$\sigma[\underline{A}] \in V \Rightarrow \sigma[\underline{A}] = \underline{A}.$$

Remark 4.11. If V is strongly generalized fluid, then V is generalized fluid.

Proposition 4.12. *Let V be a variety of algebras of type τ .*

(i) *If V is strongly generalized fluid, then for all $\underline{A} \in V$, and for all $\sigma \in P_{RG}(V)$, $\sigma[\underline{A}] = \underline{A}$.*

(ii) *If V is strongly generalized fluid, then V is generalized unsolid.*

Proof. (i) Assume that V is strongly generalized fluid. Let $\underline{A} \in V$ and $\sigma \in P_{RG}(V)$. By Proposition 4.6, we get $\sigma[\underline{A}] \in \sigma(V) \subseteq V$. Since V is strongly generalized fluid, we get $\sigma[\underline{A}] = \underline{A}$.

(ii) Assume that V is strongly generalized fluid. By (i), for all $\underline{A} \in V$ and for all $\sigma \in P_{RG}(V)$, we have $\sigma[\underline{A}] = \underline{A} = \sigma_{id}[\underline{A}]$ (i.e., $\sigma \sim_{VRG} \sigma_{id}$ and $\sigma(f_i) \approx f_i(x_1, \dots, x_{n_i}) \in IdV$ for all $i \in I$). This shows that $P_{RG}(V) \subseteq P_{RG}^0(V)$. But $P_{RG}^0(V) \subseteq P_{RG}(V)$, and then $P_{RG}(V) = P_{RG}^0(V)$. So, V is generalized unsolid. $\square$

Proposition 4.13. *If V is a strongly generalized fluid variety of algebras of type τ and $[\sigma_{id}]_{\sim_{VRG}} = [\sigma_{id}]_{\sim_{VRG-iso}}$, then V is generalized unsolid.*

Proof. Assume that V is strongly generalized fluid and $[\sigma_{id}]_{\sim_{VRG}} = [\sigma_{id}]_{\sim_{VRG-iso}}$. Let $\sigma \in P_{RG}(V)$. Since V is strongly generalized fluid, we get $\sigma[\underline{A}] \cong \underline{A}$ for all $\underline{A} \in V$ (i.e., $\sigma \sim_{VRG-iso} \sigma_{id}$). Therefore, $\sigma \in [\sigma_{id}]_{\sim_{VRG-iso}} = [\sigma_{id}]_{\sim_{VRG}}$, (i.e., $\sigma \sim_{VRG} \sigma_{id}$), and we have $\sigma \in P_{RG}^0(V)$. So $P_{RG}(V) \subseteq P_{RG}^0(V)$, but since $P_{RG}^0(V) \subseteq P_{RG}(V)$, then $P_{RG}(V) = P_{RG}^0(V)$. Therefore, V is generalized unsolid. $\square$

Proposition 4.14. *Let V be a variety of algebras of type τ . The $\sim_{VRG}|_{P_{RG}(V)}$ is a congruence relation on the monoid $(P_{RG}(V); \circ_G, \sigma_{id})$.*

Proof. Let $\sigma_1, \sigma_2 \in P_{RG}(V)$, such that $\sigma_1 \sim_{VRG}|_{P_{RG}(V)} \sigma_2$ and let $\sigma \in P_{RG}(V)$. Then, $\sigma[\underline{A}] \in V$ for all $\underline{A} \in V$.

Firstly, we will show that $\sim_{VRG}|_{P_{RG}(V)}$ is a right congruence. Since, $\sigma_1 \sim_{VRG}|_{P_{RG}(V)} \sigma_2$ implies that $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ for all $\underline{A} \in V$, and we get that $\sigma[\sigma_1[\underline{A}]] = \sigma[\sigma_2[\underline{A}]]$ since σ is a function. So, $\sigma_1 \circ_G \sigma \sim_{VRG} \sigma_2 \circ_G \sigma$, but $\sigma_1 \circ_G \sigma, \sigma_2 \circ_G \sigma \in P_{RG}(V)$, because $P_{RG}(V)$ is a monoid. Therefore, $\sigma_1 \circ_G \sigma \sim_{VRG}|_{P_{RG}(V)} \sigma_2 \circ_G \sigma$.

Next, we will show that $\sim_{VRG}|_{P_{RG}(V)}$ is a left congruence. Since, $\sigma[\underline{A}] \in V$ and $\sigma_1 \sim_{VRG}|_{P_{RG}(V)} \sigma_2$ implies that $\sigma_1[\sigma[\underline{A}]] = \sigma_2[\sigma[\underline{A}]]$. So, $\sigma \circ_G \sigma_1 \sim_{VRG} \sigma \circ_G \sigma_2$, but $\sigma \circ_G \sigma_1, \sigma \circ_G \sigma_2 \in P_{RG}(V)$, because $P_{RG}(V)$ is a monoid. Therefore, $\sigma \circ_G \sigma_1 \sim_{VRG}|_{P_{RG}(V)} \sigma \circ_G \sigma_2$. So, $\sim_{VRG}|_{P_{RG}(V)}$ is a congruence relation. $\square$

Proposition 4.15. *Let V be a variety of algebras of type τ . Then $\sim_{VRG-iso}|_{P_{RG}(V)}$ is a congruence relation on the monoid $(P_{RG}(V); \circ_G, \sigma_{id})$.*

Proof. Let $\sigma_1, \sigma_2 \in P_{RG}(V)$, such that $\sigma_1 \sim_{VRG-iso}|_{P_{RG}(V)} \sigma_2$ and let $\sigma \in P_{RG}(V)$. Then $\sigma[\underline{A}] \in V$ for all $\underline{A} \in V$.

Firstly, we will show that $\sim_{VRG}|_{P_{RG}(V)}$ is a right congruence. Since, $\sigma_1 \sim_{VRG-iso}|_{P_{RG}(V)} \sigma_2$ implies that $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ for all $\underline{A} \in V$, and by Lemma 3.12, we get that $\sigma[\sigma_1[\underline{A}]] \cong \sigma[\sigma_2[\underline{A}]]$. So, $\sigma_1 \circ_G \sigma \sim_{VRG} \sigma_2 \circ_G \sigma$, but $\sigma_1 \circ_G \sigma, \sigma_2 \circ_G \sigma \in P_{RG}(V)$, because $P_{RG}(V)$ is a monoid. Therefore, $\sigma_1 \circ_G \sigma \sim_{VRG-iso}|_{P_{RG}(V)} \sigma_2 \circ_G \sigma$.

Next, we will show that $\sim_{VRG-iso}|_{P_{RG}(V)}$ is a left congruence. Since, $\sigma[\underline{A}] \cong \sigma[\underline{A}]$ and $\sigma[\underline{A}] \in V$, then $\sigma_1[\sigma[\underline{A}]] \cong \sigma_2[\sigma[\underline{A}]]$. So, $\sigma \circ_G \sigma_1 \sim_{VRG-iso} \sigma \circ_G \sigma_2$, but $\sigma \circ_G \sigma_1, \sigma \circ_G \sigma_2 \in P_{RG}(V)$, because $P_{RG}(V)$ is a monoid. Therefore, $\sigma \circ_G \sigma_1 \sim_{VRG-iso}|_{P_{RG}(V)} \sigma \circ_G \sigma_2$. So, $\sim_{VRG-iso}|_{P_{RG}(V)}$ is a congruence relation. $\square$

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References

- [1] I. Chajda, K. Denecke and R. Halaš, Algebras Induced by Hypersubstitutions, Algebra and Discrete Mathematics, Proc. of the Conference on General Algebra and Discrete Mathematics, Potsdam 1998, Shaker Verlag (1999), 11-25.
- [2] K. Denecke, D. Lau, R. Pöschel and D. Schweigert, Hypersubstitutions, Hyperequational classes and clones congruence, Contributions to General Algebras 7 (1991), 97-118.
- [3] K. Denecke and J. Koppitz, Fluid, unsolid and completely unsolid varieties, Algebra Colloquium 7(4) (2000), 381-390.
- [4] S. Leeratanavalee and K. Denecke, Generalized Hypersubstitutions and Strongly Solid Varieties, General Algebra and Applications, Proc. of the 59th Workshop on General Algebra, 15th Conference for Young Algebraists Potsdam 2000, Shaker Verlag (2000), 135-145.
- [5] S. Leeratanavalee and S. Phatchat, Pre-strongly solid and left-edge(right-edge)-strongly solid varieties of semigroups, Int. J. Alg. 1(5) (2007), 205-226.
- [6] S. Leeratanavalee, Submonoids of generalized hypersubstitutions, Demonstratio Mathematica 40(1) (2007), 13-22.
- [7] S. Phuapong and S. Leeratanavalee, Generalized derived algebras and generalized induced algebras, to appear in Asian-European Journal of Mathematics.

